Riemannian Geometry $\boxed{\text{Week 2}}$

Recall from last time: A Riemannian metric on a smooth manifold M:

- $\forall p \in M$, $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ bilinear, symmetric, positive definite

- In any local coordinate chart $(x^1, \dots, x^n)$:

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \text{ are smooth functions}$$

If V, W are tangent vectors at p: $g(V, W) = g_{ij} V^i W^j$

Denote: $g = g_{ij}\, dx^i dx^j = g_{ij}\, dx^i \otimes dx^j$ $\quad \{dx^i \otimes dx^j\}_{i,j=1}^n$: Basis of bilinear forms on T_pM

<u>Change of coordinates</u> (proven in the exercises):

$$(x^1, \dots, x^n) \longrightarrow (y^1, \dots, y^n)$$

$$g = \tilde{g}_{ij}\, dy^i dy^j = g_{ab}\, dx^a dx^b$$

$$\tilde{g}_{ij} = \frac{\partial x^a}{\partial y^i} g_{ab} \frac{\partial x^b}{\partial y^j}$$

<u>Matrix form</u>:

$$[\tilde{g}] = \left[\frac{\partial x}{\partial y}\right]^T [g] \left[\frac{\partial x}{\partial y}\right]$$

<u>Example</u>:

- Flat space: $(\mathbb{R}^n, g_E)$, In Cartesian coordinates

$$g_E = (dx^1)^2 + \dots + (dx^n)^2$$

When $n = 2$, in polar coordinates (r, θ):

$$\left.\begin{aligned} x &= r \cdot \cos\theta \\ y &= r \cdot \sin\theta \end{aligned}\right\} \quad g_E = dr^2 + r^2 d\theta^2$$

• Hyperbolic space (half-space model):

$$\mathbb{H}^n = \{(x^1, \dots, x^n) : x^n > 0\}$$

$$g_{\mathbb{H}^n} = \frac{1}{(x^n)^2} g_E$$

Geometric properties: Those <u>invariant</u> under change of coordinates

• Norm of a tangent vector:

$$\|X\| := \sqrt{g(X,X)}$$

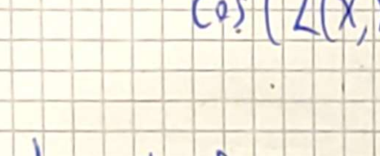
• Angle between two <u>non-zero</u> vectors:

$$\cos\left(\angle(X,Y)\right) = \frac{g(X,Y)}{\|X\| \cdot \|Y\|}$$

• <u>Length of a curve</u>:

<u>Def</u>: A curve $\gamma : (a,b) \longrightarrow M$ is called <u>piecewise C^k</u> if it is continuous and $\exists\, t_0 = a < t_1 < \dots < t_m = b$ such that

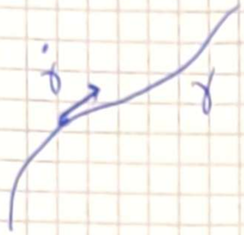
$$\gamma|_{[t_i, t_{i+1}]} \text{ is } C^k \quad \forall i = 0, \dots, m-1.$$



<u>Def</u>: Let $\gamma : (a,b) \longrightarrow (M,g)$ be piecewise C^1. Length of γ:

$$l(\gamma) := \int_a^b \|\dot\gamma\|\, dt$$

$\longrightarrow$ Makes sense on $\bigcup_{i=0}^{m-1} (t_i, t_{i+1})$



Recall: tangent vector of γ:

$$\dot{\gamma}(t)(f) = \frac{d}{ds}\left(f \circ \gamma(s)\right)\Big|_{s=t}$$

In local coordinates: $\gamma : t \longrightarrow (\gamma^1(t), \dots, \gamma^n(t))$

$$\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$$

__Lemma__: The length functional has the following properties:

1) $\ell(\gamma) \geq 0$, "=" iff γ is constant.

2) Additivity: If $\gamma : [a,b] \to M$, $c \in (a,b)$:

$$\ell(\gamma|_{[a,b]}) = \ell(\gamma|_{[a,c]}) + \ell(\gamma|_{[c,b]}).$$

3) Invariant under reparametrization:

If $h : (c,d) \longrightarrow (a,b)$ is piecewise C^1, monotone and onto, then

$$\ell(\gamma \circ h) = \ell(\gamma) \qquad \text{(see Exercises)}$$

We can endow (M,g) with the structure of a __metric space__:

__Def__: __Riemannian distance function__:

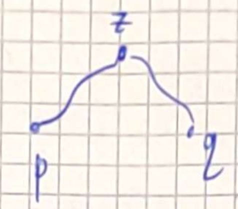
Let (M,g) be a Riemannian manifold; $p, q \in M$. We define $d_g : M \times M \longrightarrow [0, +\infty]$ by

$$d_g(p,q) = \inf \left\{ \ell(\gamma) : \gamma : [0,1] \to M, \text{ piecewise } C^1, \ \gamma(0) = p, \ \gamma(1) = q \right\}$$

Properties

- $d_g(p,q) = +\infty$ iff p, q belong to different connected components of M.

- (M, d_g) is a metric space
  - $d_g(p,q) = d_g(q,p)$
  - $d_g(p,q) \leqslant d_g(p,z) + d_g(z,q)$



  - $d_g(p,q) \geqslant 0$, " = " iff $p = q$. (see exercises)

Riemannian volume:

Let (M,g) be a Riem. manifold and $h \in C^0(M)$. We want to define in a geometric (i.e. coordinate independent) way the integral
$$\int_M h \, dV_g$$

Def: Let (U, ϕ) be a local coordinate chart, with associated coordinates $(x^1, \dots, x^n)$
$$\int_U h \, dV_g := \int_{\phi(U) \subseteq \mathbb{R}^n} h(\phi^{-1}(x)) \cdot \sqrt{\det [g_{ij}(x)]} \, dx^1 \dots dx^n$$

Lem: The above definition is coordinate independent.

Proof: If $(y^1, \dots, y^n)$ is a different coordinate chart associated to a $\tilde{\phi}$, then:

- In the new coordinates:

$$\det[\tilde{g}] = (\det[\frac{\partial x}{\partial y}])^2 \det[g]$$

- If $F: \Omega_{\subseteq \mathbb{R}^n} \xrightarrow{x \longrightarrow y} \tilde{\Omega}_{\subseteq \mathbb{R}^n}$ is a diffeo:

$$\int_\Omega h(x)\,dx = \int_{\tilde{\Omega}} h(F^{-1}(y)) \left|\det\left(\frac{\partial x}{\partial y}\right)\right| dy$$

$$\frac{\partial x}{\partial y} = \left(\frac{\partial F}{\partial x}\right)^{-1}$$

So: With $F = \tilde{\phi} \circ \phi^{-1}$. If U the common domain of $\phi, \tilde{\phi}$.

$$\int_{\phi(U)} h(\phi^{-1}(x)) \sqrt{\det[g]}\,dx = \int_{\tilde{\phi}(U) = F(\phi(U))} h(\underbrace{\phi^{-1}(F^{-1}(y))}_{\tilde{\phi}^{-1}(y)}) \underbrace{\sqrt{\det g}\, \left|\det\left(\frac{\partial x}{\partial y}\right)\right|}_{\sqrt{\det \tilde{g}}} dy$$

$\square$

In order to define $\int_M h\,dV_g$: Let $\{\chi_a\}_{a \in A}$ be a partition of unity, such that $\operatorname{supp} \chi_a$ is contained in some local coordinate chart $\forall a \in A$.

Then $h = \sum_{a \in A} \chi_a \cdot h$. Define $\int_M h\,dV_g = \sum_{a \in A} \int_M h \cdot \chi_a\,dV_g$

• Volume of $U \subseteq M$: $\int_U dV_g$

• If M is oriented: Volume form (n-form)

in local coordinates: $\omega_g = \sqrt{\det[g]}\, dx^1 \wedge \cdots \wedge dx^n$

Invariant under changes of coordinates $x \to y$ with $\det\left[\frac{\partial y}{\partial x}\right] > 0$.

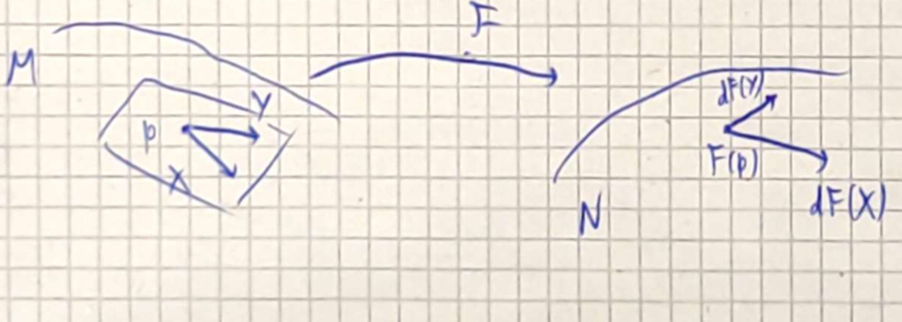
<u>Pull-back of a metric:</u>

Let $F: M \longrightarrow (N, g)$ be an <u>immersion</u>. We will define the <u>pull-back</u> of g via F as follows:

$\forall p \in M$, $\forall X, Y \in T_p M$:

push-forward

$$F^* g(X, Y) := g(dF_p(X), dF_p(Y))$$



<u>Recall</u>: In local coordinates $(x^1, \dots, x^m)$ on M, $(y^1, \dots, y^n)$ on N:

$$(dF(X))^a = \frac{\partial F^a}{\partial x^i} X^i$$

Lemma: F^*g is a Riemannian metric

Pf: It is obviously symmetric - bilinear.

Positivity: $F^*g(X,X) = g\left(dF(X), dF(X)\right) \geq 0$

$$\cdots = 0 \Leftrightarrow dF(X) = 0 \overset{F:\text{ immersion}}{\Longleftrightarrow} X = 0. \quad \blacksquare$$

We also say that F^*g is the metric induced on M by F.

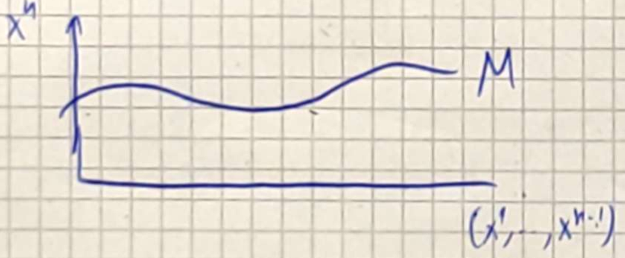
In local coordinates:

$$F^*g_{ij} = g\left(dF\left(\tfrac{\partial}{\partial x^i}\right), dF\left(\tfrac{\partial}{\partial x^j}\right)\right)$$

$$= \frac{\partial F^a}{\partial x^i} \cdot g_{ab} \cdot \frac{\partial F^b}{\partial x^j}$$

Example: Let M be the graph of a function

$$f: \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \quad \text{in } \mathbb{R}^n$$



So: Local coordinates $(x^1, \ldots, x^{n-1})$ on M (from $\mathbb{R}^{n-1}$)

f defines an immersion

$$F(x^1, \ldots, x^{n-1}) = (x^1, \ldots, x^{n-1}, f(x^1, \ldots, x^{n-1}))$$

So $$\frac{\partial F}{\partial x^i} = (0, \ldots, 0, \underset{i\text{-th}}{\underbrace{1}}, 0, \ldots, 0, \frac{\partial f}{\partial x^i})$$

Induced metric

$$(F^* g_E)_{ij} = g_E\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) = \delta_{ij} + \frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j}$$

<u>Alternative way:</u>

$$F^i(x^1, \dots, x^{n-1}) = x^i, \quad i = 1, \dots, n-1$$

$$F^n(x^1, \dots, x^{n-1}) = f(x^1, \dots, x^{n-1})$$

So: $y^i = F^i(x)$

$$g_E = (dy^1)^2 + \dots + (dy^n)^2$$

$$F^* dy^i = dF^i(x) = \frac{\partial F^i}{\partial x^j} dx^j$$

and $$F^* g_E = (dF^1(x))^2 + \dots + (dF^n(x))^2$$

$$= (dx^1)^2 + \dots + (dx^n)^2 + \left(\frac{\partial f}{\partial x^i} dx^i\right)^2$$

$$= \delta_{ij}\, dx^i dx^j + \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} dx^i dx^j \quad \square$$

<u>Definition</u>: Let $F: (M,g) \longrightarrow (N,h)$ be a smooth map.

- We will say that F is a <u>local</u> isometry if $dF_p: (T_pM, g_p) \longrightarrow (T_{F(p)}N, h)$ is a (linear) isometry $\forall p \in M$

Implies that dF is bijective
$\Rightarrow$ F is an immersion and locally a diffeo
$\Rightarrow$ $F^*h = g$

- We will say that F is a global isometry if it is a local isometry and bijective

(In this case: (M,g) and (N,h) are practically identical geometrically)

- F is (locally) conformal if dF_p is bijective $\forall p$ and $F^*h = \lambda \cdot g$ for some $\lambda \in C^\infty(M)$, $\lambda > 0$

- Local isometries preserve $\|\cdot\|$, angles
- Locally conformal maps preserve <u>angles</u>

Ex: Consider $(\mathbb{R}^n, g_E)$

and $(\mathbb{T}^n, g_{\mathbb{T}^n} = (dx^1)^2 + \dots + (dx^n)^2)$

$$\mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n$$

Then $F: \mathbb{R}^n \to \mathbb{T}^n$, $x \to x \bmod 2\pi\mathbb{Z}^n$

is a local isometry (but not global)

Lemma: If $F: (M,g) \to (N,h)$ is ~~[illegible]~~ an isometry of metric spaces $\Longrightarrow$ F is smooth and a Riemannian isometry.

Theorem: If M is a smooth manifold, then it admits a Riemannian metric.

Proof: Let $\{U_\alpha, \varphi_\alpha\}_{\alpha\in A}$ be an atlas on M and let $\{\chi_\beta\}_{\beta\in B}$ be a subordinate partition of unity, i.e:

$\forall \beta \in B$: $\chi_\beta \in C^\infty(M)$, $\chi_\beta : M \to [0,1]$, $\operatorname{supp}\chi_\beta \subseteq U_{\alpha(\beta)}$ for some $\alpha(\beta) \in A$, $\operatorname{supp}\chi_{\beta_1} \cap \operatorname{supp}\chi_{\beta_2} \neq \emptyset$ for finitely many β_2,

$\sum_\beta \chi_\beta = 1$.

$\forall \beta$: On $U_{\alpha(\beta)}$ consider the metric $g_\beta = \varphi^*_{\alpha(\beta)} g_E$

(in the local coordinates: $g_\beta = (dx^1)^2 + \dots + (dx^n)^2$

Set $g = \sum_\beta \chi_\beta \cdot g_\beta$

- g well-defined: Finite sum at every point
- $\forall p \in M$: g symmetric, bilinear,

$$g(X,X) \geq 0 \quad \forall X \in T_pM$$

and $g(X,X) = 0 \iff \sum_\beta \chi_\beta \cdot g_\beta(X,X) = 0 \overset{\chi_\beta \geq 0}{\Longrightarrow}$

$$\Rightarrow \forall \beta: \chi_\beta(p) \cdot g_\beta(X,X) = 0$$

$$\exists \beta \text{ s.t. } \chi_\beta(p) \neq 0 \Rightarrow g_\beta(X,X) = 0 \rightarrow X = 0.$$

$\blacksquare$